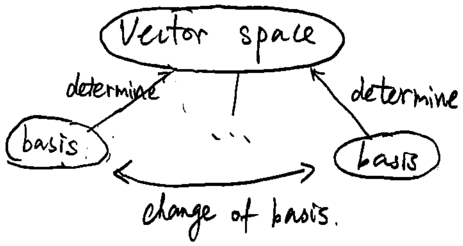
# Tutorial 2 Basis and Dimension

(The "proofs" are sketchy. You are NOT supposed to write like this in exam!).

## Linear independence, span, and basis

1. Vector spaces are very regularly behaved because of the notion of basis. We understand the whole vector space if we understand the basis. <u>Everytime you have trouble proving something, use a basis!</u>




Upshot 2.1. Vector space V.S. basis.

- Basis uniquely determines vector space.
- Vector space can have many basis, they are related by change of basis.

2. Basis are special to vector spaces! They have too many properties that is not at all common for other algebraic structures. Basis are well-behaved.

Prop. 2.2. Let $V$ be a finite dimensional vector space.

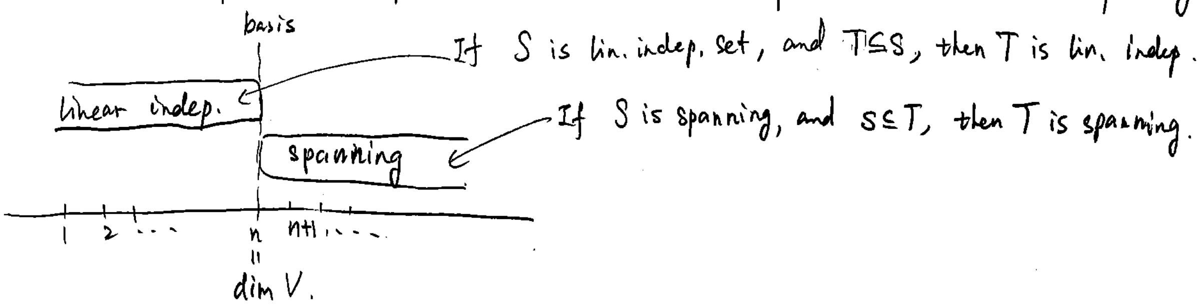
i). (existence). Every $V$ has a basis.

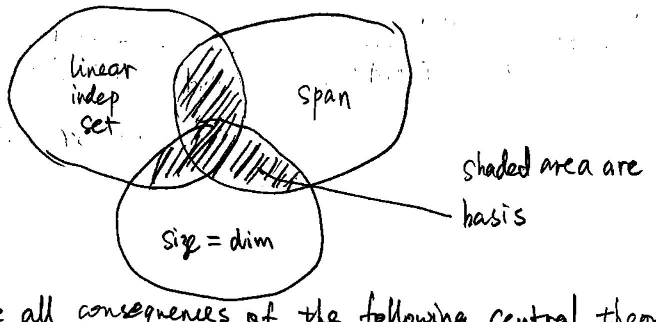
ii). (extension). Every linearly indep. set can be extended to a basis.

iii). (reduction) Every spanning set can be reduced to a basis.

iv). The dimension, i.e., the size of basis, is well-defined for $V$. (All basis has same size).

Basis = linear independence + span = maximal linear independent set = minimal spanning set.




knowing 2 implies the remaining 1.

They are all consequences of the following central theorem.

Thm 2.3. (Replacement theorem).

Let $V$ be a finite dimensional vector space. If

$S_1 = \{v_1, \cdots, v_n\} \subseteq V$ : linear independent set.

$S_2 = \{w_1, \cdots, w_m\} \subseteq V$ : spanning set.

then $|S_1| \leq |S_2|$.

[The proof uses a replacement argument, explaining the name of thm.]

"pf." Initial step: As $v_1 \in V = \text{span}\{w_1, \cdots, w_m\}$, we have $w_1 \in \text{span}\{v_1, w_2, w_3, \cdots, w_m\} = V$.

(Replace $w_1$ by $v_1$). (Because $v_1 = \sum_{i=1}^{m} a_i w_i \Rightarrow w_1 = a_1^{-1}\left(\sum_{i=2}^{m} a_i w_i - v_1\right) \in \text{span}\{v_1, w_2, w_3, \cdots, w_n\}$)

$\Rightarrow \text{span}\{v_1, w_2, w_3, \cdots, w_m\} = \text{span}\{v_1, w_1, w_2, w_3, \cdots\} = V$ ).

Induction step. For $1 < i \leq \min\{n, m\}$, we are in situation

(Replace $w_i$ by $v_i$)

$$\text{span}\{\underbrace{v_1, \cdots, v_{i-1}}_{\text{linearly indep.}}, w_i, \cdots, w_n\} = V.$$

$v_i \in \text{span}\{v_1, \cdots, v_{i-1}, w_i, \cdots, w_n\}$, and $v_i \notin \text{span}\{v_1, \cdots, v_{i-1}\}$.

so $\exists\, w_j \in \{w_i, \cdots, w_n\}$ ; WLOG $w_i$, s.t. $w_i \in \text{span}\{v_1, \cdots, v_{i-1}, v_i, w_{i+1}, \cdots, w_m\}$.

If $m < n$, then we are in situation

$\text{span}\{v_1, \cdots, v_m\} = V$, but $v_{m+1} \notin \text{span}\{v_1, \cdots, v_m\} = V$ contradition. $\square$.

Q1. If $\{v_1, v_2, v_3\}$ is a basis, then $\{v_1+v_2, v_2+v_3, v_3\}$ is also a basis.

(The same proof work for any $M \in GL_3(\mathbb{F})$, $M\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is also a basis).

pf. (lin. indep.) $0 = (a_1\ a_2\ a_3)\begin{pmatrix} v_1+v_2 \\ v_2+v_3 \\ v_3 \end{pmatrix}$

$$= (a_1\ a_2\ a_3)\left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right]$$

$$\overset{\text{why?}}{=} \left[(a_1\ a_2\ a_3)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= (a_1\ \ a_1+a_2\ \ a_2+a_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\xRightarrow[\text{of } v_1, v_2, v_3]{\text{lin. indep}} (0,0,0) = (a_1\ \ a_1+a_2\ \ a_2+a_3) = (a_1\ a_2\ a_3)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

As $\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \neq 0$; $(a_1\ a_2\ a_3) = (0,0,0)$ by multiplying inverses on both sizes.

(span). One may check $v_1, v_2, v_3 \in \mathrm{Span}\{v_1+v_2, v_2+v_3, v_3\}$ respectively. But we will do it in a systematic way.

$$
\begin{aligned}
&(a_1\ \ a_2\ \ a_3)\begin{pmatrix} v_1+v_2 \\ v_2+v_3 \\ v_3 \end{pmatrix} \\
&= (a_1\ \ a_2\ \ a_3)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \Rightarrow \mathrm{span}\{v_1+v_2, v_2+v_3, v_3\} \subseteq \mathrm{span}\{v_1, v_2, v_3\}. \\
&= (a_1\ \ a_1+a_2\ \ a_2+a_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.
\end{aligned}
$$

$$\quad\ \ \| \qquad\quad \| \qquad\quad //$$

so $$(b_1\ \ b_2\ \ b_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (b_1\ \ b_2\ \ b_3)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} v_1+v_2 \\ v_2+v_3 \\ v_3 \end{pmatrix}. \Rightarrow \mathrm{span}\{v_1, v_2, v_3\} \subseteq \mathrm{span}\left\{\begin{matrix} v_1+v_2, \\ v_2+v_3, \\ v_3 \end{matrix}\right\}.$$

$\square$.

Q2. Recall the inclusion-exclusion principle.

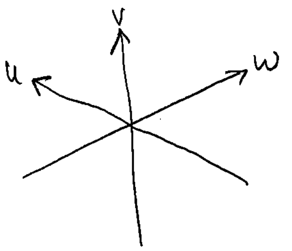
For finite dim. vector subspaces $U, V$ of $W$.

$$\dim(U+V) = \dim U + \dim V - \dim(U\cap V).$$

This is always true.

Prove or give a counterexample: For f.d. vector subspaces $U, V, W$ of $W$.

$$\dim(U+V+W) = \dim U + \dim V + \dim W - \dim(U\cap V) - \dim(U\cap W) - \dim(V\cap W) + \dim(U\cap V\cap W).$$




Where is the problem?



Ans: $(U+V)\cap W \neq U\cap W + V\cap W$.